EFFECT OF A SMALL VISCOSITY ON THE POTENTIAL FLOW OF A LIQUID WITH A FREE BOUNDARY IN THE FORM OF AN ELLIPSE

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UDC 532.516.2

We study the effect of a small viscosity on plane potential flow of a liquid with a free boundary in the form of the ellipse derived in [1]. Suppose at zero time the liquid with the velocity field

$$v_x = \sqrt{2}x/2,\ v_y = -\sqrt{2}y/2 \tag{1}$$

is contained in the circle $x^2 + y^2 \leq 1$ whose boundary is a free surface. The pressure and tangential stress on the free boundary S_t are zero for all $t \geq 0$, and there are no external forces. The corresponding flow of an ideal incompressible liquid is potential and has the form [1]

$$\begin{gathered} v_x = \dot{\tau}\tau^{-1}x,\ v_y = -\dot{\tau}\tau^{-1}y, \\ p = -0{,}5\tau\ddot{\tau}\,(x^2\tau^{-2} + y^2\tau^2 - 1), \\ \int_1^\tau \sqrt{1+\rho^{-4}}\,d\rho = \lambda t\,(\lambda = \text{const}),\ \tau(0) = 1,\ \dot{\tau} = d\tau/dt. \end{gathered} \tag{2}$$

The solution of (2) can be interpreted as follows. As t increases the free boundary $x^2 + y^2 = 1$ is deformed into the ellipse L_t: $x^2\tau^{-2} + y^2\tau^2 = 1$ with semiaxes $\tau(t)$ and $\tau^{-1}(t)$. It follows from (2) that $\tau \to \infty$ and $\tau^{-1} \to 0$ for $t \to \infty$ and $\lambda > 0$. The ellipse is drawn out along the Ox axis. If $\lambda < 0$, $\tau \to 0$ as $t \to \infty$ and the ellipse is drawn out along the Oy axis.

For vanishing viscosity ($\nu \to 0$) a boundary layer is formed close to the free boundary S_t in which the derivatives of the velocity vary rapidly and a finite vorticity appears. Everywhere outside the boundary layer region the behavior of the viscous liquid is similar to that of an ideal liquid.

The flow of a viscous incompressible liquid is described by the Navier–Stokes equations

$$\partial \mathbf{v}/\partial t + (\mathbf{v}, \nabla)\mathbf{v} = -\nabla p + \varepsilon^2\Delta\mathbf{v},\ \operatorname{div}\mathbf{v} = 0\ (\varepsilon^2 = 1/\mathrm{Re}) \tag{3}$$

with the initial conditions (1) and the kinematic and dynamic conditions on the free boundary S_t [2]

$$\partial F/\partial t + \mathbf{v}\cdot\nabla F = 0; \tag{4}$$

$$4n_xn_y\frac{\partial v_x}{\partial x} + \left(n_y^2 - n_x^2\right)\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right) = 0; \tag{5}$$

$$p - 2\varepsilon^2\left[n_x^2\frac{\partial v_x}{\partial x} + n_y^2\frac{\partial v_y}{\partial y} + n_xn_y\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right)\right] = 0. \tag{6}$$

Here $F(x, y, t) = 0$ is the equation of the free boundary S_t in implicit form, $\mathbf{n} = (n_x, n_y)$ is a unit vector along the inward normal to the free boundary S_t, and Re is the Reynolds number. The quantities in (3)-(6) are dimensionless.

Rostov-on-Don. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 1, pp. 101-105, January-February, 1977. Original article submitted March 15, 1976.

Problem (3)-(6) is solved by the asymptotic boundary-layer method [3]. The asymptotic expansions of the solution of the problem as $\varepsilon \to 0$ are constructed in the form

$$\mathbf{v} \sim \sum_{k=0}^{N} \varepsilon^k \mathbf{v}_k(x, y, t) + \sum_{k=0}^{N+1} \varepsilon^k \mathbf{h}_k(x, y, t, \varepsilon);$$
$$p \sim \sum_{k=0}^{N} \varepsilon^k p_k(x, y, t) + \sum_{k=0}^{N+1} \varepsilon^k q_k(x, y, t, \varepsilon); \quad (7)$$
$$\tilde{\zeta} \sim \sum_{k=0}^{N+1} \varepsilon^k \tilde{\zeta}_k(x, t),\ \mathbf{v}_k = (v_{xk}, v_{yk}),\ \mathbf{h}_k = (h_{xk}, h_{yk}),$$

where $y = \tilde{\zeta}(x, t)$ is the equation of the free boundary S_t.

The functions $\mathbf{v}_0$ and p_0 describe the flow of an ideal incompressible liquid with the free boundary L_t and the initial condition (1); their values are determined from equations given in [2].

The functions $\mathbf{v}_k$ and p_k are found in the first iteration procedure [3] and satisfy linear equations of the form

$$\frac{\partial \mathbf{v}_k}{\partial t} + \sum_{i+j=k} (\mathbf{v}_i, \nabla) \mathbf{v}_j = -\nabla p_k + \Delta \mathbf{v}_{k-2},$$
$$\operatorname{div} \mathbf{v}_k = 0,\ \mathbf{v}_k|_{t=0} = 0\ (\mathbf{v}_{-1} \equiv 0,\ k \geqslant 1). \quad (8)$$

Since rot $\mathbf{v}_0 = 0$ it follows from (8) that rot $\mathbf{v}_k = 0$. Introducing Φ_k by the relation $\mathbf{v}_k = \operatorname{grad} \Phi_k$ we obtain from (8) the equations for Φ_k and p_k:

$$\Delta \Phi_k = 0,$$
$$\frac{\partial \Phi_k}{\partial t} + \dot{\tau}\tau^{-1} x \frac{\partial \Phi_k}{\partial x} - \dot{\tau}\tau^{-1} y \frac{\partial \Phi_k}{\partial y} + p_k = -\frac{1}{2} \sum_{i=1}^{k-1} \mathbf{v}_i \mathbf{v}_{k-i}. \quad (9)$$

The boundary-layer functions $\mathbf{h}_k$ and q_k are concentrated in the neighborhood of the free boundary S_t and compensate the discrepancies in $\mathbf{v}_k$ and p_k in satisfying the dynamic condition (5). We construct the functions $\mathbf{h}_k$ and q_k by introducing moving local coordinates (ρ, φ) close to the boundary L_t by the expression

$$x = \tau(1 - \rho\tau^{-2}\delta^{-1})\cos\varphi,\ y = \tau^{-1}(1 - \rho\tau^{2}\delta^{-1})\sin\varphi,$$
$$\delta = \sqrt{\tau^2 \sin^2\varphi + \tau^{-2}\cos^2\varphi},\ \varphi \in [0, 2\pi],$$

where $x = \tau \cos\alpha$ and $y = \tau^{-1} \sin\alpha$ are the parametric equations of the ellipse L_t, ρ is the distance from the point (x, y) to L_t, and φ is the value of the parameter α corresponding to the point on L_t closest to (x, y).

Let us determine the equations satisfied by the functions $\mathbf{h}_k$ and q_k. Let $h_{\rho k}$, $h_{\varphi k}$, $v_{\rho k}$, and $v_{\varphi k}$, be, respectively, the components of the vectors $\mathbf{h}_k$ and $\mathbf{v}_k$ in the coordinates (ρ, φ). Substituting (7) into (3) and using (8) and (2) we write the equations obtained in local coordinates. We expand the known coefficients in Taylor series in powers of ρ and take account of the relation $\partial\rho/\partial t + \mathbf{v}_0 \cdot \nabla\rho = 0$ which is valid at $\rho = 0$ and expresses the property that the boundary L_t be a liquid contour for all $t \geq 0$. We set $\rho = \varepsilon s$ and equate the coefficients of $\varepsilon^0, \varepsilon^1, \ldots, \varepsilon^N$ to zero. As shown in (4), $\mathbf{h}_0 = q_0 = h_{\rho 1} = q_1 = 0$. To determine $\mathbf{h}_k$ and q_k we obtain the equations

$$\partial h_{\varphi k}/\partial t + s a(t, \varphi) \partial h_{\varphi k}/\partial s - a(t, \varphi) h_{\varphi k} = \partial^2 h_{\varphi k}/\partial s^2 + F_{k-1},$$
$$\partial q_{k+1}/\partial s = -2\dot{\tau}\tau^{-1}\delta^{-2} \sin 2\varphi\, h_{\varphi k} + M_{k-1}, \quad (10)$$
$$\frac{\partial h_{\rho,k+1}}{\partial s} = \sum_{n=0}^{k} s^n \delta^{-(3n+1)} \left(\delta^{-2} h_{\rho,k-n} - \frac{\partial h_{\varphi,k-n}}{\partial \varphi}\right),$$
$$\mathbf{h}_k|_{t=0} = 0,\ \mathbf{h}_k|_{s=\infty} = 0,\ q_k|_{s=\infty} = 0,\ F_0 = M_0 = 0.$$

Here $a(t, \varphi) = \dot{\tau}\tau^{-1}\delta^{-2}(\tau^{-2}\cos^2\varphi - \tau^2\sin^2\varphi)$, and F_{k-1} and M_{k-1} are known and are expressed in terms of $\mathbf{v}_0, \ldots, \mathbf{v}_k$; $\mathbf{h}_0, \ldots, \mathbf{h}_{k-1}$.

Similarly by applying the first and second iteration processes simultaneously to the dynamic condition (5) we obtain the boundary conditions for $h_{\varphi k}$ in (10) for $s = 0$:

$$\partial h_{\varphi k}/\partial s = -(\delta^{-1}\partial v_{\rho,\, k-1}/\partial\varphi + \partial v_{\varphi,\, k-1}/\partial\rho + \delta^{-3}v_{\varphi,\, k-1}) + Q_{k-1} \quad (\rho = 0),\ Q_0 = 0. \tag{11}$$

To determine $h_{\varphi 1}$ we set $k = 1$ in (10) and (11), introduce a new function $H = \delta h_{\varphi 1}$, and make the change of variables $\xi = \delta(t, \varphi)s$, $t_1 = t$. Finally by defining a variable

$$\beta = \int_0^t \delta^2(t, \varphi)\, dt,$$

we obtain for $H(\xi, \varphi, \beta)$ the problem

$$\partial H/\partial\beta = \partial^2 H/\partial\xi^2,$$
$$H|_{\beta=0} = 0, \quad H|_{\xi=\infty} = 0, \quad \partial H/\partial\xi = \psi(\beta, \varphi)\ (\xi = 0),$$

where $\psi(\beta, \varphi)$ is the value of the function $2\dot{\tau}\tau^{-1}\delta^{-2}\sin 2\varphi$ in the variables (β, φ). The solution of the last problem has the following form in the old variables:

$$h_{\varphi_1} = 2\delta^{-1}\pi^{-1/2}\sin 2\varphi \int_0^t \frac{\dot{\tau}(u)}{\tau(u)}[\beta(t, \varphi) - \beta(u, \varphi)]^{-1/2} \exp\left[-\frac{\delta^2(t, \varphi)s^2/4}{\beta(t, \varphi) - \beta(u, \varphi)}\right] du.$$

We find from (10)

$$q_2 = 4\dot{\tau}\tau^{-1}\delta^{-4}\sin^2 2\varphi \int_0^t \frac{\dot{\tau}(u)}{\tau(u)} \operatorname{erfc}\left[\frac{s\delta(t, \varphi)}{2\sqrt{\beta(t, \varphi) - \beta(u, \varphi)}}\right] du.$$

We next determine the equations satisfied by the functions $\zeta_k(t, \varphi)$. Let $\rho = \zeta(t, \varphi, \varepsilon) \sim \sum_{k=0}^{N} \varepsilon^k \zeta_k(t, \varphi)$ be the equation of the free boundary S_t in local coordinates; here $\zeta_0 = 0$, since $\rho = 0$ is the equation of L_t. We set $F = -\rho + \zeta(t, \varphi, \varepsilon)$ and by applying the first and second iteration procedures simultaneously to (4) [3] we obtain

$$\begin{gathered} \partial\zeta_k/\partial t - a(t, \varphi)\zeta_k = [h_{\rho k} + v_{\rho k}]_{\rho=0} + N_{k-1}, \\ \zeta_k|_{t=0} = 0,\ N_0 = N_1 = 0\ (k \geqslant 1). \end{gathered} \tag{12}$$

Proceeding similarly with the dynamic conditions (6) on S_t we obtain the boundary conditions for systems (9) on L_t:

$$\begin{gathered} p_k + q_k + \delta\tau\ddot{\tau}\zeta_k = 2\partial v_{\rho,\, k-2}/\partial\rho + D_{k-1} \quad (\rho = 0), \\ D_0 = D_1 = 0 \quad (k \geqslant 1). \end{gathered} \tag{13}$$

As shown in [4], $p_1 = \zeta_1 = \mathbf{v}_1 = 0$, and N_{k-1} and D_{k-1} are known.

We now set $k = 2$ in (9), (12), and (13), eliminate p_2, and introduce the function $\eta = \delta\zeta_2$. To determine Φ_2 and η we obtain the following problem in the ellipse $D_t(x^2\tau^{-2} + y^2\tau^2 \le 1)$:

$$\begin{gathered} \Delta\Phi_2 = 0, \\ \partial\Phi_2/\partial t - \tau\ddot{\tau}\eta = 4\dot{\tau}\tau^{-1}\delta^{-4}\sin^2 2\varphi \ln\tau - 2a(t, \varphi) \quad (\rho = 0), \\ \partial\eta/\partial t - \delta\partial\Phi_2/\partial\rho = 4\delta^{-4}(\tau^{-2}\cos^2\varphi - \tau^2\sin^2\varphi) \quad (\rho = 0), \\ \eta = \Phi_2 = 0 \quad (t = 0). \end{gathered} \tag{14}$$

In the domain D_t we transform to elliptical coordinates (σ, θ): $x = c\cosh\sigma\cos\theta$, $y = c\sinh\sigma\sin\theta$ $(\sigma \ge 0,\ 0 \le \theta \le 2\pi)$, where $\tau = c\cosh\sigma_0$ and $\tau^{-1} = c\sinh\sigma_0$ are the semiaxes of the ellipse, and $\sigma = \sigma_0$ is the equation of the contour L_t. We expand Φ_2 and η in series

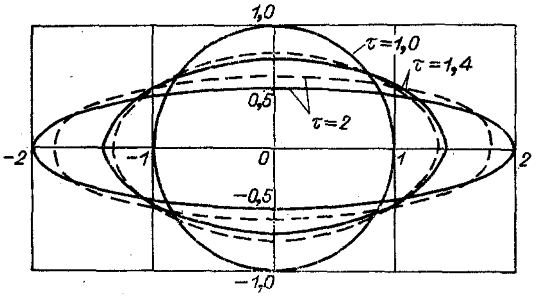

Fig. 1

$$\Phi_2 = \sum_{k=0}^{\infty} \omega_k(\tau) \frac{\operatorname{ch} k\sigma}{\operatorname{ch} k\sigma_0} \cos k\theta, \quad \eta = \sum_{k=0}^{\infty} \eta_k(\tau) \cos k\varphi.$$

From (14) we obtain the system of linear equations

$$\frac{\tau^2}{\sqrt{1+\tau^4}} \frac{d\omega_k}{d\tau} - \frac{2\tau^4}{(1+\tau^2)^2} \eta_k = A_k(\tau),$$
$$\frac{\tau^2}{\sqrt{1+\tau^4}} \frac{d\eta_k}{d\tau} + kc_k\omega_k = B_k(\tau),$$
$$\omega_k = \eta_k = 0 \qquad (\tau = 1)$$

for the coefficients ω_k and η_k.

The coefficients A_k, B_k, c_k are known:

$$A_0 = -\frac{2\tau}{(1+\tau^2)\sqrt{1+\tau^4}} \left(1 - \tau^2 - \frac{4\tau^2 \ln \tau}{1+\tau^2}\right),$$
$$A_{2k} = -\frac{8\tau^3 (\tau^2-1)^{k-2}}{\sqrt{1+\tau^4}\,(1+\tau^2)^{k+1}} \left(\tau^2 - 1 - 2\frac{\tau^4 - 2k\tau^2 + 1}{\tau^2+1} \ln \tau\right),$$

$$B_{2k} = \frac{16k\tau^2 \ln \tau}{(1+\tau^2)^2} \left(\frac{\tau^2-1}{\tau^2+1}\right)^{k-1}, \quad A_{2k+1} = B_{2k+1} = 0 \quad (k \geqslant 0),$$
$$c_0 = c_{-2} = 0, \; c_2 = \frac{2\tau^2}{1+\tau^4}; \; c_{2k} = \frac{c_2 + c_{2k-2}}{1 + c_2 c_{2k-2}} \qquad (k \geqslant 1).$$

The last system was solved numerically on an M-222 computer by the Runge–Kutta method. The form of the free boundary is shown in Fig. 1 for τ = 1, τ = 1.4, and τ = 2. The solid curve represents the boundary L_t and the open curve S_t. Whether the ellipse is drawn out or flattened in the course of time the effect of viscosity is to slow down the process and to "round" the free boundary.

The author thanks L. S. Srubshchik and V. I. Yudovich for posing the problem and for a discussion of the results.

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